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INTERACTION OF TWO ELLIPTIC INCLUSIONS

MARK J. MEISNER
Department of Civil Engineering, University of Arizona, Arizona, U.S.A.

and

DEMITRIS A. KOURIS
Department of Mechanical and Aerospace Engineering, Arizona State University, Arizona, U.S.A.

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Abstract—The paper presents the solution of a boundary value problem involving two interacting elliptical inhomogeneities in an infinite elastic body. The loading cases considered include a far-field biaxial tension and a thermally induced residual field that is modeled by uniform eigenstrains sustained by the inhomogeneities. The mathematical formulation is based upon the Papkovich–Neuber displacement approach. Certain interesting aspects of the solution and the effects of perfectly bonded and slipping interfaces are discussed in some detail.

STATEMENT OF THE PROBLEM AND THE MATHEMATICAL MODEL

Consider an infinite region with two elliptic inhomogeneities, Ω_1 and Ω_2, with centers at O_1 and O_2, respectively. Let the centers be at the origins of the Cartesian coordinates (x_1, y_1) and (x_2, y_2), and the x_1, x_2-axis be the center line, as illustrated in Fig. 1. If the central distance $O_1O_2 = \zeta$, then

$$x_1 = x_2 + \zeta, \quad y_1 = y_2. \tag{1}$$

It is convenient to use a coordinate system such that the boundaries involved in the problem correspond to a constant value of one coordinate. The elliptical coordinates (α, β) will be used in the present investigation.

The elliptic coordinates are obtained from the coordinate transformation

$$x_i = C \cosh \alpha_i \cos \beta_i, \quad y_i = C \sinh \alpha_i \sin \beta_i, \tag{2}$$

where $i = 1, 2$ and C is the eccentricity of the inclusions.

The total displacement vector $\mathbf{u}$ may be represented by

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2, \tag{3}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are the displacement vectors corresponding to the coordinate centers, O_1 and O_2, respectively.

Utilizing the Papkovich–Neuber displacement formulation (Papkovich, 1932; Neuber, 1934), the displacement fields, $\mathbf{u}_1$ and $\mathbf{u}_2$, are given by

$$2G\mathbf{u}_1 = \text{grad}\,[\varphi_{01} + x_1\varphi_{11} + y_1\varphi_{21}] - (\kappa+1)[\varphi_{11}, \varphi_{21}] \tag{4}$$

and

$$2G\mathbf{u}_2 = \text{grad}\,[\varphi_{02} + x_2\varphi_{12} + y_2\varphi_{22}] - (\kappa+1)[\varphi_{12}, \varphi_{22}], \tag{5}$$

where G is the shear modulus, κ is the Kolosov constant, and φ_{ij} are arbitrary harmonic

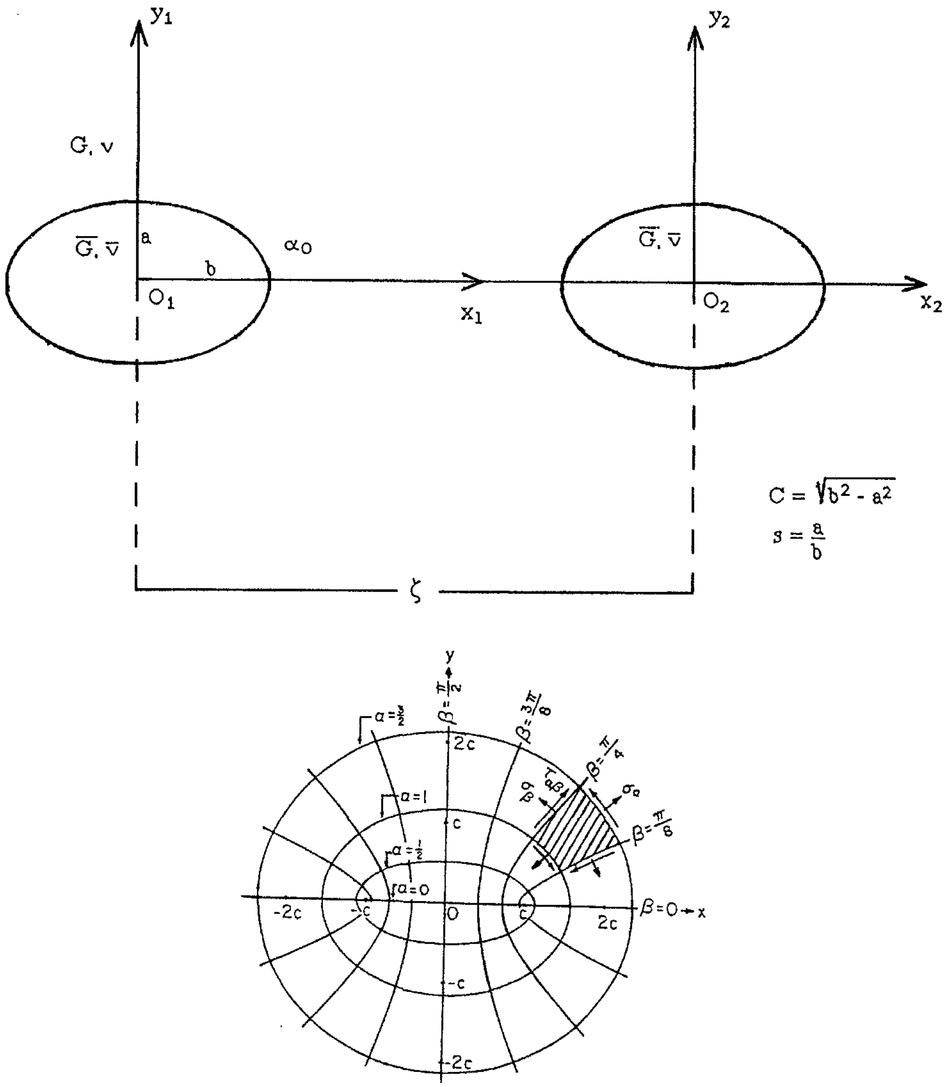

Fig. 1. Geometry of the problem and the elliptical coordinate system.

functions. $[\varphi_{ij}, \varphi_{mn}]$ corresponds to φ_{ij} for the x-component and φ_{mn} for the y-component of the displacement vector. In order to satisfy the boundary conditions along Ω_1 ($\alpha_1 = \alpha_0$), it is necessary to express eqn (5) in terms of the O_1 coordinate system. Substitution of eqn (1) into eqn (5) leads to

$$2G\mathbf{u}_2 = \operatorname{grad}[\varphi_{02} - \zeta\varphi_{12} + x_1\varphi_{12} + y_1\varphi_{22}] - (\kappa+1)[\varphi_{12}, \varphi_{22}]. \tag{6}$$

For the boundary conditions along Ω_2 ($\alpha_2 = \alpha_0$), a similar procedure yields

$$2G\mathbf{u}_1 = \operatorname{grad}[\varphi_{01} + \zeta\varphi_{11} + x_2\varphi_{11} + y_2\varphi_{21}] - (\kappa+1)[\varphi_{11}, \varphi_{21}]. \tag{7}$$

Thermal loading

The residual field due to thermal loading is modeled by a pair of uniform eigenstrains (ε_x^*, ε_y^*). These eigenstrains (Mura, 1987) are proportional to the temperature change and the mismatch of the inclusion/matrix thermal expansion coefficients. Consequently, residual displacements are introduced in the inhomogeneities according to

$$\bar{u}_x^* = \varepsilon_x^* x,$$
$$\bar{u}_y^* = \varepsilon_y^* y. \tag{8}$$

Their corresponding components in elliptical coordinates are

$$\bar{u}_\alpha^* = \frac{C^2 h}{4}\sinh 2\alpha[(1+\cos 2\beta)\varepsilon_x^* + (1-\cos 2\beta)\varepsilon_y^*],$$
$$\bar{u}_\beta^* = \frac{C^2 h}{4}\sin 2\beta[(1+\cosh 2\alpha)\varepsilon_x^* + (1-\cosh 2\alpha)\varepsilon_y^*], \tag{9}$$

where

$$h = \frac{1}{C(\cosh^2\alpha - \cos^2\beta)^{1/2}}. \tag{10}$$

When the inhomogeneities are perfectly bonded, the boundary conditions along the elliptical interfaces ($\alpha_i = \alpha_0$) are

$$u_\alpha = \bar{u}_\alpha + \bar{u}_\alpha^*, \quad u_\beta = \bar{u}_\beta + \bar{u}_\beta^*, \quad \sigma_\alpha = \bar{\sigma}_\alpha, \quad \text{and} \quad \tau_{\alpha\beta} = \bar{\tau}_{\alpha\beta}. \tag{11}$$

Here, $(u_\alpha, u_\beta, \sigma_\alpha, \tau_{\alpha\beta})$ denote elastic quantities in the matrix, $(\bar{u}_\alpha, \bar{u}_\beta, \bar{\sigma}_\alpha, \bar{\tau}_{\alpha\beta})$ denote elastic quantities in the inclusion, and $(\bar{u}_\alpha^*, \bar{u}_\beta^*)$ indicate the contribution of the thermal field.

In the case of a slipping interface, the boundary conditions become

$$u_\alpha = \bar{u}_\alpha + \bar{u}_\alpha^*, \quad \sigma_\alpha = \bar{\sigma}_\alpha, \quad \tau_{\alpha\beta} = \bar{\tau}_{\alpha\beta} = 0, \tag{12}$$

due to the assumption of perfect slip, which implies that no shear tractions can be sustained by the interface.

Biaxial tension

In the case of biaxial tension (T_x, T_y) at infinity, the applied load is represented by the following potential set:

$$\varphi_0 = \frac{1}{8}(T_x - T_y)(\kappa+1)(x^2 - y^2),$$
$$\varphi_1 = -\frac{1}{2}T_y x,$$
$$\varphi_2 = -\frac{1}{2}T_x y. \tag{13}$$

The boundary conditions along the two interfaces are given by

$$u_\alpha + u_\alpha^0 = \bar{u}_\alpha, \quad u_\beta + u_\beta^0 = \bar{u}_\beta, \quad \sigma_\alpha + \sigma_\alpha^0 = \bar{\sigma}_\alpha, \quad \text{and} \quad \tau_{\alpha\beta} + \tau_{\alpha\beta}^0 = \bar{\tau}_{\alpha\beta} \quad \text{(perfect bonding)}, \tag{14}$$

and by

$$u_\alpha + u_\alpha^0 = \bar{u}_\alpha, \quad \sigma_\alpha + \sigma_\alpha^0 = \bar{\sigma}_\alpha, \quad \tau_{\alpha\beta} + \tau_{\alpha\beta}^0 = \bar{\tau}_{\alpha\beta} = 0 \quad \text{(perfect slip)}. \tag{15}$$

The quantities $u_\alpha^0, u_\beta^0, \sigma_\alpha^0$, and $\tau_{\alpha\beta}^0$ denote the contribution of the applied tension and are derived from the potential functions given in eqn (13).

SOLUTION OF THE BOUNDARY VALUE PROBLEM

If a set of harmonic displacement potentials can be determined such that the boundary conditions along the interfaces and the far field are satisfied, a unique solution can be obtained. Based upon geometric considerations, the Papkovich–Neuber displacement potentials for the matrix ($\alpha_i > \alpha_0$) are chosen as

$$\begin{aligned}
\varphi_{0i} &= p_0 \left[F_0^i \alpha_i + \sum_{n=1}^{\infty} A_n^i \, e^{-n\alpha_i} \cos n\beta_i \right] \\
\varphi_{1i} &= p_0 \sum_{n=1}^{\infty} B_n^i \, e^{-n\alpha_i} \cos n\beta_i \\
\varphi_{2i} &= 0,
\end{aligned} \tag{16}$$

where p_0 is a normalizing factor related to the applied load (thermal or mechanical). Similarly, the displacement potentials for the two inhomogeneities ($\alpha_i < \alpha_0$) are chosen as

$$\begin{aligned}
\bar{\varphi}_{0i} &= p_0 \sum_{n=1}^{\infty} \bar{A}_n^i \cosh n\alpha_i \cos n\beta_i \\
\bar{\varphi}_{1i} &= p_0 \sum_{n=1}^{\infty} \bar{B}_n^i \cosh n\alpha_i \cos n\beta_i \\
\bar{\varphi}_{2i} &= 0.
\end{aligned} \tag{17}$$

The potential functions given in eqns (16)–(17) represent the disturbance of the elastic field in the matrix and the inclusions, respectively, due to the presence of the inclusions. As a result, their contribution decays away from the inhomogeneities. The boundary conditions along the two elliptical interfaces provide the means for the evaluation of the unknown coefficients (i.e. $F_0^i, A_n^i, B_n^i, \bar{A}_n^i$ and $\bar{B}_n^i$).

By enforcing the boundary conditions, eight equations are obtained for the interfaces $\Omega_1(\alpha_1 = \alpha_0)$ and $\Omega_2(\alpha_2 = \alpha_0)$. However, since the inhomogeneities are identical and a symmetric load is applied, only one of the interfaces needs to be considered, provided that the relations presented in Appendix A are appropriately utilized and that the following symmetry relations among the coefficients are taken into account when $\alpha_1 = \alpha_2 = \alpha_0$:

$$\begin{aligned}
F_0 &= F_0^1 = F_0^2 \\
A_n &= A_n^1 = (-1)^n A_n^2, \qquad \bar{A}_n = \bar{A}_n^1 = (-1)^n \bar{A}_n^2 \\
B_n &= B_n^1 = (-1)^{n+1} B_n^2, \qquad \bar{B}_n = \bar{B}_n^1 = (-1)^{n+1} \bar{B}_n^2.
\end{aligned} \tag{18}$$

The boundary conditions in the case of eigenstrain loading are presented below; the equations are identical to the ones corresponding to the mechanical load, with the exception of the forcing terms.

Consequently, the continuity of the normal displacement given by $u_\alpha = \bar{u}_\alpha + \bar{u}_v^*$ at $\alpha_1 = \alpha_0$ yields

$$F_0\delta_{n,0}-A_nU_{A1}-\frac{C}{4}[B_{n-1}U_{B1}+B_{n+1}U_{B2}]$$

$$+(1-\delta_{n,1})\left[F_0Z_n+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n}\right]U_{C1}$$

$$-\frac{C}{4}\sum_{m=1}^{\infty}B_m[(1-\delta_{n,1})d_{m,n-1}U_{C2}+(1-\delta_{n,0})d_{m,n+1}U_{C3}]$$

$$-\frac{1}{\Gamma}[\bar{A}_n\bar{U}_{A1}+\frac{C}{4}(\bar{B}_{n-1}\bar{U}_{B1}+\bar{B}_{n+1}\bar{U}_{B2})]$$

$$=\frac{C^2}{4}\sinh 2\alpha_0[(\varepsilon_x^*+\varepsilon_y^*)\delta_{n,0}+(\varepsilon_x^*-\varepsilon_y^*)\delta_{n,2}]\quad (n=0,1,2,\ldots). \tag{19}$$

For the displacements in the tangential direction, $u_\beta = \bar{u}_\beta+\bar{u}_\beta^*$ at $\alpha_1 = \alpha_0$ gives

$$A_nV_{A1}+\frac{C}{4}[B_{n-1}V_{B1}+B_{n+1}V_{B2}]+(1-\delta_{n,1})\left[F_0Z_n+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n}\right]V_{C1}$$

$$-\frac{C}{4}\sum_{m=1}^{\infty}B_m[(1-\delta_{n,1})d_{m,n-1}V_{C2}+d_{m,n+1}V_{C3}]-\frac{1}{\Gamma}[\bar{A}_n\bar{V}_{A1}+\frac{C}{4}(\bar{B}_{n-1}\bar{V}_{B1}+\bar{B}_{n+1}\bar{V}_{B2})]$$

$$=\frac{C^2}{4}[(\varepsilon_x^*+\varepsilon_y^*)+(\varepsilon_x^*-\varepsilon_y^*)\cosh 2\alpha_0]\delta_{n,2}\quad (n=1,2,\ldots). \tag{20}$$

where $\Gamma = \bar{G}/G$.

The requirement for continuity of normal and shear tractions yields

$$-\frac{1}{2}F_0\sinh 2\alpha_0\delta_{n,0}-\frac{1}{4}[A_{n-2}S_{A1}-A_nS_{A2}+A_{n+2}S_{A3}]$$

$$-\frac{C}{16}[B_{n-3}S_{B1}-B_{n-1}S_{B2}-B_{n+1}S_{B3}+B_{n+3}S_{B4}]$$

$$-\frac{1}{4}\left[F_0Z_{n-2}+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n-2}\right]S_{C1}+\frac{1}{4}\left[F_0Z_n+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n}\right]S_{C2}$$

$$-\frac{1}{4}\left[F_0Z_{n+2}+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n+2}\right]S_{C3}$$

$$+\frac{C}{16}\sum_{m=1}^{\infty}B_m[d_{m,n-3}S_{C4}-d_{m,n-1}S_{C5}-d_{m,n+1}S_{C6}+d_{m,n+3}S_{C7}]$$

$$+\frac{1}{4}[\bar{A}_{n-2}\bar{S}_{A1}-\bar{A}_n\bar{S}_{A2}+\bar{A}_{n+2}\bar{S}_{A3}]$$

$$+\frac{C}{16}[\bar{B}_{n-3}\bar{S}_{B1}-\bar{B}_{n-1}\bar{S}_{B2}-\bar{B}_{n+1}\bar{S}_{B3}+\bar{B}_{n+3}\bar{S}_{B4}]=0\quad (n=0,1,2,3,..) \tag{21}$$

and

$$
\begin{aligned}
&-\frac{1}{2}F_0\delta_{n,2}-\frac{1}{4}[A_{n-2}T_{A1}-A_nT_{A2}+A_{n+2}T_{A3}]\\
&-\frac{C}{16}[B_{n-3}T_{B1}-B_{n-1}T_{B2}-B_{n+1}T_{B3}+B_{n+3}T_{B4}]\\
&+\frac{1}{4}\left[F_0Z_{n-2}+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n-2}\right]T_{C1}-\frac{1}{4}\left[F_0Z_n+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n}\right]T_{C2}\\
&+\frac{1}{4}\left[F_0Z_{n+2}+\sum_{m=1}^{\infty}(A_m+\zeta B_m)d_{m,n+2}\right]T_{C3}\\
&-\frac{C}{16}\sum_{m=1}^{\infty}B_m[d_{m,n-3}T_{C4}-d_{m,n-1}T_{C5}-d_{m,n+1}T_{C6}+d_{m,n+3}T_{C7}]\\
&-\frac{1}{4}[\bar{A}_{n-2}\bar{T}_{A1}-\bar{A}_n\bar{T}_{A2}+\bar{A}_{n+2}\bar{T}_{A3}]\\
&-\frac{C}{16}[\bar{B}_{n-3}\bar{T}_{B1}-\bar{B}_{n-1}\bar{T}_{B2}-\bar{B}_{n+1}\bar{T}_{B3}+\bar{B}_{n+3}\bar{T}_{B4}]=0\quad(n=1,2,3,..),
\end{aligned}
\tag{22}
$$

respectively. $\delta_{i,j}$ denotes Kronecker's delta; U_{ij}, V_{ij}, S_{ij} and T_{ij} are known functions of n, α_0, κ and $\bar{\kappa}$ (Appendix B).

After solving the system of eqns (19)–(22) for the unknown coefficients F_0, A_n, B_n, $\bar{A}_n$, and $\bar{B}_n$, stresses and displacements anywhere in the elastic body can be determined. The series converge as n increases and no more than 10 terms are required for matching the boundary conditions to a level of three significant figures. The mathematical formulation for the case of perfect slip is similar to the one presented above [using the conditions (12) or (15)].

RESULTS AND DISCUSSION

Based upon the formulation presented above, the solution can be obtained for any combination of the eigenstrains (ε_x^*, ε_y^*) and the far-field tension (T_x, T_y).

Results are presented for the cases of uniform thermal expansion $\varepsilon_x^* = \varepsilon_y^*$ and all-around tension $T_x = T_y$. Poisson's ratios of the matrix and the inhomogeneities have been assumed equal ($\nu = \bar{\nu} = 0.3$).

Variations of the inclusion shape, shear moduli ratio and relative distance between the inhomogeneities are introduced through the parameters s, Γ and λ, respectively. These dimensionless parameters are defined by

$$s=\frac{a}{b},\quad \Gamma=\frac{\bar{G}}{G}\quad\text{and}\quad \lambda=\frac{\zeta}{2b}. \tag{23}$$

Figures 2–5 illustrate the results in the case of uniform eigenstrains $\varepsilon_x^* = \varepsilon_y^*$. By keeping the aspect and shear moduli ratios constant, the dependence of the normal stress σ_α along the interface on the inclusion relative distance can be determined (Fig. 2). As the relative distance λ decreases, the stress concentration increases, particularly along the slipping interface. For values of λ greater than 2.5 (separation distance is greater than three times the major semi-axis of the elliptic inhomogeneity), the interaction effects are no longer present and the solution for the single inclusion in an infinite medium is recovered. In analyzing the normal stress along the interfaces, variations of the relative stiffness Γ ($\bar{G}/G$) have also been considered. As the inclusions become stiffer relative to the matrix (Γ increases), the absolute values of the matrix interfacial stresses increase. In both cases (perfect bond and sliding) the normal stress σ_α assumes an absolute maximum at $\beta = 0$, with the higher value corresponding to the case of perfect slip. The effect of the shear

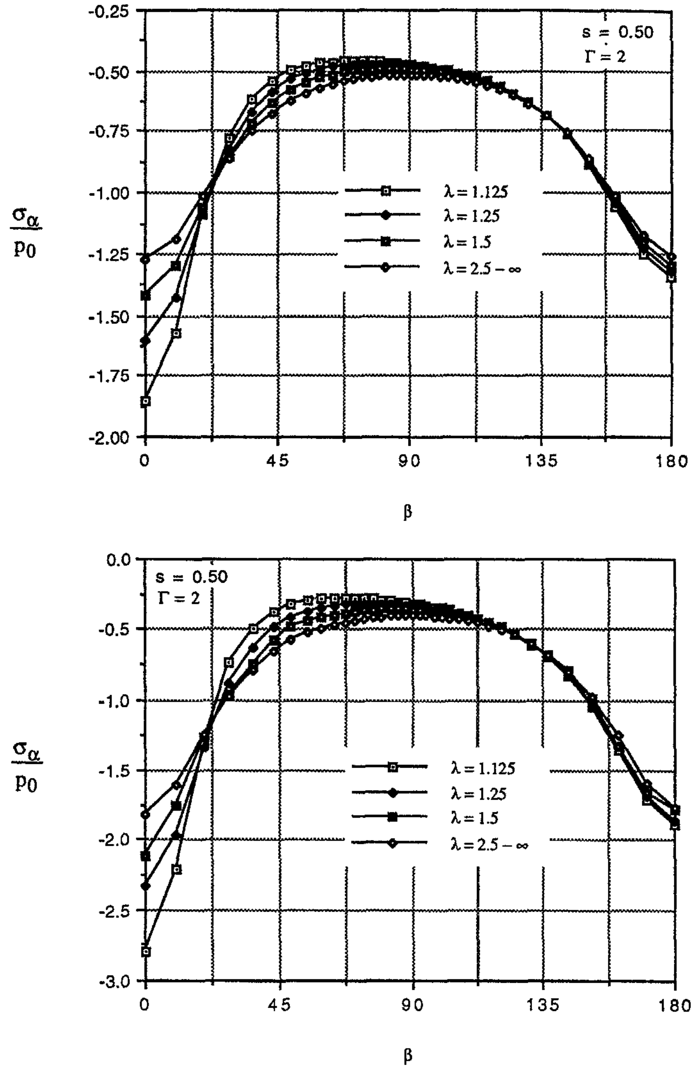

Fig. 2. Eigenstrain loading: normal stress along the interface as a function of the relative distance between the inclusions for (a) perfect bonding, (b) sliding.

moduli ratio Γ (relative stiffness) on the hoop stress σ_β is more pronounced in the case of perfect bonding. Nevertheless, higher stress concentrations are again obtained at $\beta = 0$. When the interface is free to slip, the hoop stress σ_β shown in Fig. 3 increases considerably at $\beta = 0$ with decreasing λ.

The stress in the y-direction along the inclusion central line is shown in Figs 4 and 5. In the first case (Fig. 4), the relative stiffness varies while the aspect and distance ratios are kept constant. It is clear that the variation of σ_y in the inclusions and the matrix, as well as the discontinuity at the interface, are dependent upon the condition of the interface (bonded vs slipping interface). The examination of the aspect ratio effect yields the stress distribution shown in Fig. 5. In the case of perfect slip along the interface, the discontinuity of σ_y at the interface increases drastically as $s \to 0$. The results for $s = 0.99$ ($a \to b = 2$), correspond to the solution of two circular inclusions given by Kouris and Tsuchida (1991).

Finally, the effects of various shear moduli and aspect ratios on the interfacial matrix stresses are investigated. It is found that the normal matrix stress σ_x at the interface remains compressive for all values of the relative distance λ between the inclusions. This is not the case, however, for the normal stress σ_y. It can be observed that the distributions of σ_y for perfect bonding and sliding are drastically different. When the effect of different aspect

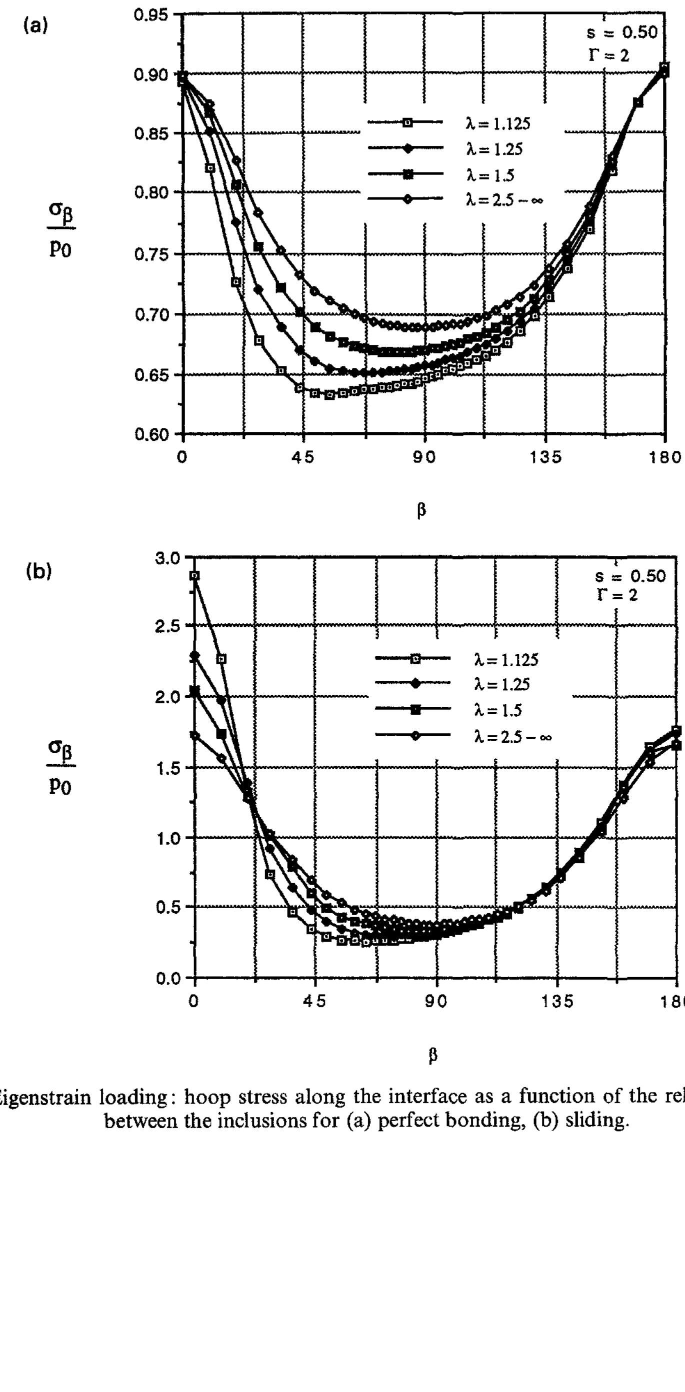

Fig. 3. Eigenstrain loading: hoop stress along the interface as a function of the relative distance between the inclusions for (a) perfect bonding, (b) sliding.

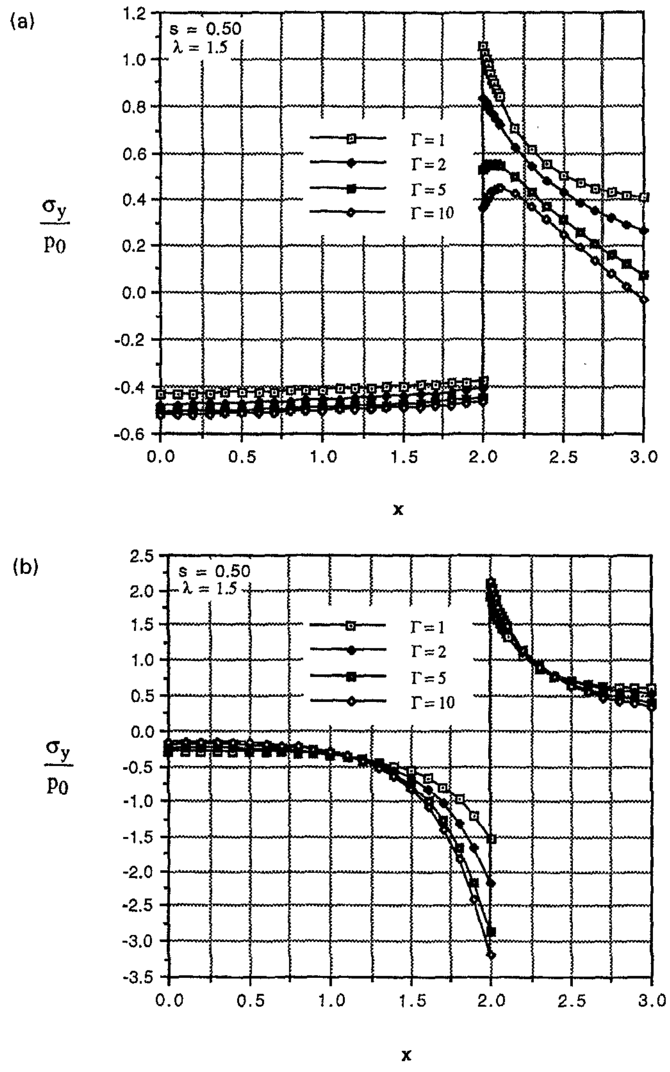

Fig. 4. Eigenstrain loading: stress distribution along the inclusion central line for various Γ, (a) perfect bonding, (b) sliding.

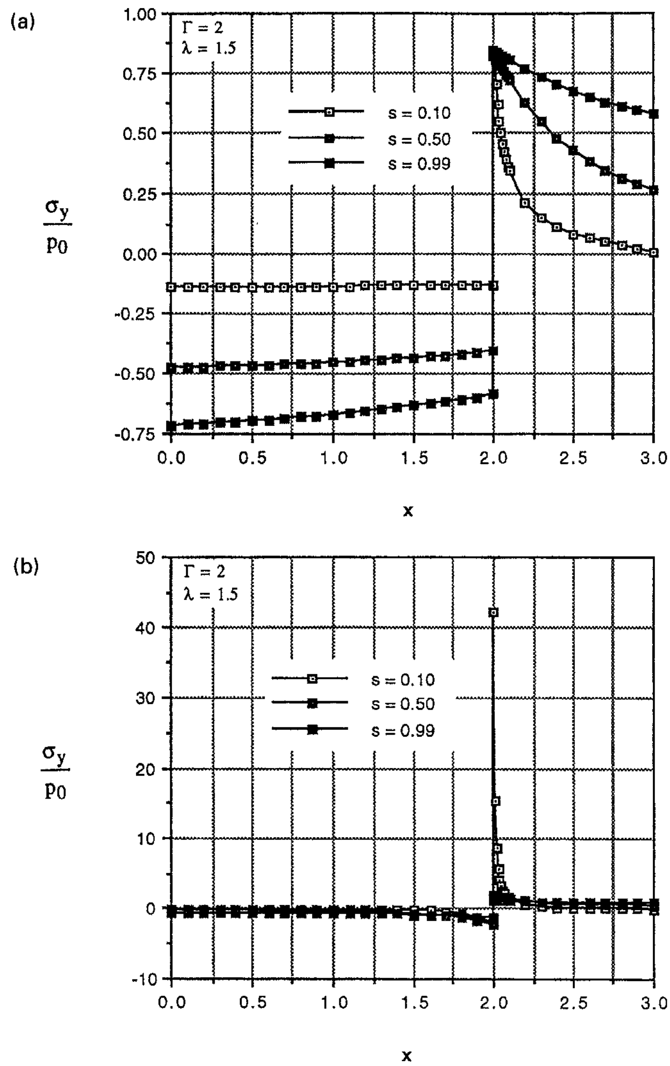

Fig. 5. Eigenstrain loading: stress distribution along the inclusion central line as a function of the inclusion shape, (a) perfect bonding, (b) sliding.

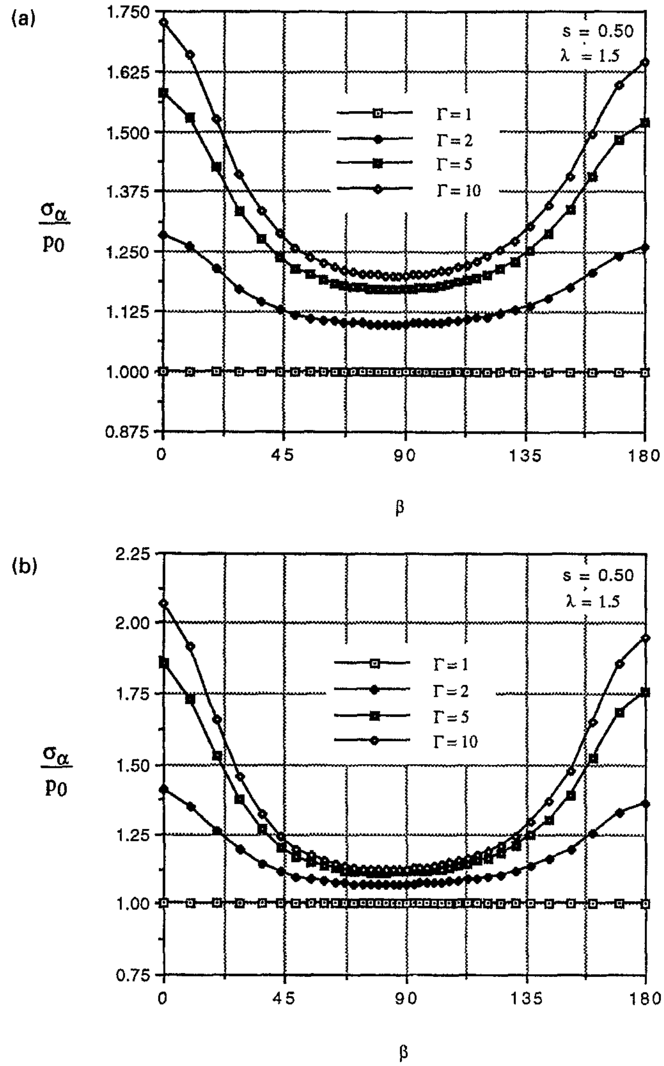

Fig. 6. Biaxial tension: normal stress along the interface as a function of Γ for (a) perfect bonding, (b) sliding.

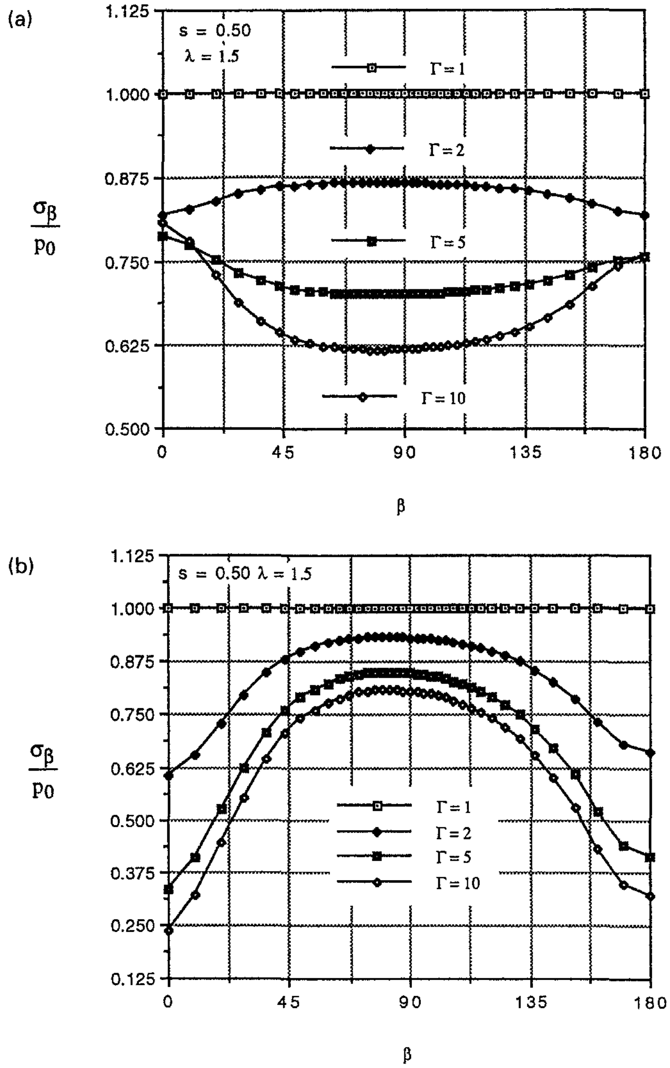

Fig. 7. Biaxial tension: hoop stress along the interface as a function of Γ for (a) perfect bonding, (b) sliding.

ratios is considered, a loss of the interfacial bonding corresponds to very high tensile values of σ_y. As the value of s decreases, the differences between perfect bonding and sliding become more pronounced.

In the case of all-around tension ($T_x = T_y$), the concentration of the normal stress σ_α along the interface is proportional to the relative stiffness Γ (Fig. 6). However, the opposite is true in the case of the matrix stress σ_β (Fig. 7). As the relative distance between the inhomogeneities decreases, the matrix stresses σ_α and σ_β increase (Fig. 8).

CONCLUSIONS

The present study analyzes the interaction of two interacting inhomogeneities subjected to thermo-mechanical loading. The problem was formulated using the Papkovich–Neuber displacement potentials and analytical solutions were obtained for the cases of perfectly bonded and slipping interfaces.

Unlike Eshelby's (1957) result for the single inclusion, the stress field inside the interacting inhomogeneities is no longer uniform. The local elastic field is determined in

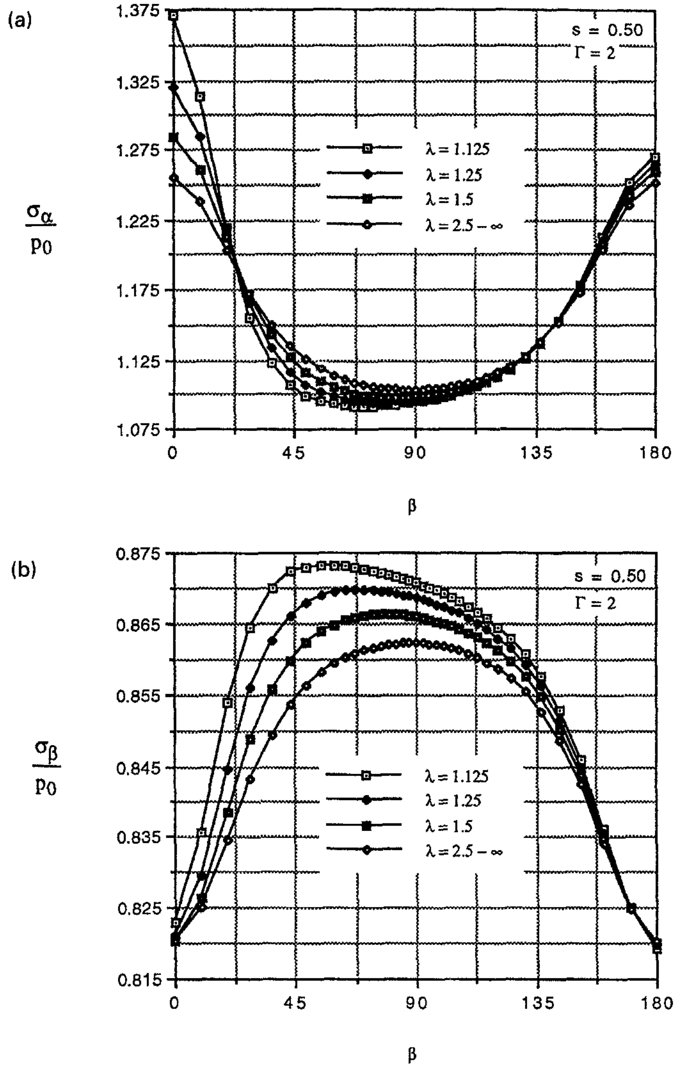

Fig. 8. Biaxial tension: normal and hoop stresses along the interface as a function of the relative distance between the inclusions in the case of perfect bonding.

the form of infinite series and is dependent upon the relative distance, the aspect ratio, and the elastic properties of the inhomogeneities. In addition, loss of the interfacial bond yields high stress concentrations.

A number of special cases can be explored by considering the appropriate limits of the shear moduli ratio Γ and the aspect ratio s. As s approaches unity, the solution for two circular inclusions are obtained. In the case of $s \to 0$, the solution corresponds to the problem of two thin, interacting, inserts in an infinite body. The crack and anticrack solutions can then be obtained by setting $\Gamma \to 0$ and $\Gamma \to \infty$, respectively.

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APPENDIX A

The following relations between the elliptic harmonic functions are utilized in the expressions for the boundary conditions along $\Omega_1(\alpha_1 = \alpha_0)$ and $\Omega_2(\alpha_2 = \alpha_0)$:

$$\alpha_1 = \sum_{n=0}^{\infty} \omega_n \cosh n\alpha_2 \cos n\beta_2,$$

$$\alpha_2 = \sum_{n=0}^{\infty} Z_n \cosh n\alpha_1 \cos n\beta_1,$$

$$e^{-m\alpha_1} \cos m\beta_1 = \sum_{n=0}^{\infty} (-1)^n d_{m,n} \cosh n\alpha_2 \cos n\beta_2,$$

$$e^{-m\alpha_2} \cos m\beta_2 = \sum_{n=0}^{\infty} (-1)^m d_{m,n} \cosh n\alpha_1 \cos n\beta_1. \tag{A1}$$

The coefficients ω_n and Z_n are given by

$$\omega_0 = Z_0 = \alpha_d + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} d_{2m,0} \tag{A2}$$

and

$$Z_n = \omega_n(-1)^n = -\frac{2}{n} e^{-n\alpha_d} + \sum_{m=1}^{\infty} \frac{(-1)^m}{m} d_{2m,n}$$

where $n = 1, 2, \ldots$, and

$$\alpha_d = \cosh^{-1}\left(\frac{\zeta}{C}\right). \tag{A3}$$

The coefficients $d_{m,n}$ can be obtained by starting with (Cooke, 1959)

$$e^{-m\alpha_1} \cos m\beta_1 = m \int_0^{\infty} I_m(\lambda c) \lambda^{-1} e^{-\lambda x} 1 \cos \lambda y_1 \, d\lambda \tag{A4}$$

and

$$e^{-\lambda x_2} \cos \lambda y_2 = \sum_{n=0}^{\infty} \varepsilon_n (-1)^n I_n(\lambda C) \cosh n\alpha_2 \cos n\beta_2, \tag{A5}$$

where $x_2 > 0$, $m = 1, 2, \ldots$, and

$$\varepsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n = 1, 2, \ldots \end{cases} \tag{A6}$$

From the relations (1), eqns (A5) and (A6) can be transformed into

$$e^{-m\alpha_1} \cos m\beta_1 = m \int_0^{\infty} I_m(\lambda C) \lambda^{-1} e^{-\lambda\zeta} \sum_{n=0}^{\infty} \varepsilon_n (-1)^n I_n(\lambda C) \cosh n\alpha_2 \cos n\beta_2 \, d\lambda. \tag{A7}$$

According to the definition of $d_{m,n}$ given in eqn (A1), the coefficients $d_{m,n}$ are obtained by

$$d_{m,n} = \varepsilon_n m \int_0^\infty I_m(\lambda C) I_n(\lambda C)\, e^{-\lambda\zeta} \lambda^{-1}\, d\lambda. \tag{A8}$$

The evaluation of the integral in eqn (A8) is discussed in Erdelyi (1954).

APPENDIX B

The functions U_{ij}, V_{ij}, S_{ij}, and T_{ij} are the following:

$$U_{A1} = n\, e^{-n\alpha_0}, \quad U_{B1} = (n-1+\kappa)\, e^{-(n-2)\alpha_0} + (n-1-\kappa)\, e^{-n\alpha_0},$$
$$U_{B2} = (n+1+\kappa)\, e^{-n\alpha_0} + (n+1-\kappa)\, e^{-(n+2)\alpha_0}, \quad U_{C1} = n \sinh n\alpha_0,$$
$$U_{C2} = (n-1+\kappa) \sinh (n-2)\alpha_0 + (n-1-\kappa) \sinh n\alpha_0,$$
$$\bar{U}_{A1} = n \sinh n\alpha_0, \quad U_{C3} = (n+1+\kappa) \sinh n\alpha_0 + (n+1-\kappa) \sinh (n+2)\alpha_0,$$
$$\bar{U}_{B1} = (n-1+\bar{\kappa}) \sinh (n-2)\alpha_0 + (n-1-\bar{\kappa}) \sinh n\alpha_0,$$
$$\bar{U}_{B2} = (n+1+\bar{\kappa}) \sinh n\alpha_0 + (n+1-\bar{\kappa}) \sinh (n+2)\alpha_0,$$

$$V_{A1} = n\, e^{-n\alpha_0}, \quad V_{B1} = (n-1-\kappa)\{e^{-(n-2)\alpha_0} + e^{-n\alpha_0}\},$$
$$V_{B2} = (n+1+\kappa)\{e^{-n\alpha_0} + e^{-(n+2)\alpha_0}\}, \quad V_{C1} = n \cosh n\alpha_0,$$
$$V_{C2} = (n-1-\kappa)\{\cosh (n-2)\alpha_0 + \cosh n\alpha_0\},$$
$$V_{C3} = (n+1+\kappa)\{\cosh n\alpha_0 + \cosh (n+2)\alpha_0\}, \quad \bar{V}_{A1} = n \cosh n\alpha_0,$$
$$\bar{V}_{B1} = (n-1-\bar{\kappa})\{\cosh (n-2)\alpha_0 + \cosh n\alpha_0\},$$
$$\bar{V}_{B2} = (n+1+\bar{\kappa})\{\cosh n\alpha_0 + \cosh (n+2)\alpha_0\},$$

$$S_{A1} = (n-2)(n-3)\, e^{-(n-2)\alpha_0},$$
$$S_{A2} = n\{(n+1)\, e^{-(n-2)\alpha_0} + (n-1)\, e^{-(n+2)\alpha_0}\} - 2\, e^{\alpha_0}\delta_{n,1},$$
$$S_{A3} = (n+2)(n+3)\, e^{-(n+2)\alpha_0},$$
$$S_{B1} = (n-3)\{(n-2)\, e^{-(n-4)\alpha_0} + (n-3-\kappa)\, e^{-(n-2)\alpha_0}\},$$
$$S_{B2} = (n-1)\{(n+2)\, e^{-(n-4)\alpha_0} - 2(\kappa-2)\, e^{-(n-2)\alpha_0}$$
$$+ 4e^{-n\alpha_0} + (n-1-\kappa)\, e^{-(n+2)\alpha_0}\} - (\kappa+1)\delta_{n,2},$$
$$\bar{S}_{B3} = (n+1)\{(n+1+\kappa)\, e^{-(n-2)\alpha_0} - 4\, e^{-n\alpha_0} + 2\,(\kappa-2)\, e^{-(n+2)\alpha_0} + (n-2)\, e^{-(n+4)\alpha_0}\}$$
$$-2\{(\kappa+1)\, e^{-\alpha_0} + e^{-3\alpha_0}\}\delta_{n,1},$$
$$\bar{S}_{B4} = (n+3)\{(n+3+\kappa)\, e^{-(n+2)\alpha_0} + (n+2)\, e^{-(n+4)\alpha_0}\},$$
$$S_{C1} = (n-2)(n-3) \cosh (n-2)\alpha_0,$$
$$S_{C2} = n\{(n+1) \cosh (n-2)\alpha_0 + (n-1) \cosh (n+2)\alpha_0\},$$
$$S_{C3} = (n+2)(n+3) \cosh (n+2)\alpha_0,$$
$$S_{C4} = (n-3)\{(n-2) \cosh (n-4)\alpha_0 + (n-3-\kappa) \cosh (n-2)\alpha_0\},$$
$$S_{C5} = (n-1)\{(n+2) \cosh (n-4)\alpha_0 - 2(\kappa-2) \cosh (n-2)\alpha_0$$
$$+ 4 \cosh n\alpha_0 + (n-1-\kappa) \cosh (n+2)\alpha_0\} - (\kappa+1)\delta_{n,2},$$
$$S_{C6} = (n+1)\{(n+1+\kappa) \cosh (n-2)\alpha_0 - 4 \cosh n\alpha_0 + 2(\kappa-2) \cosh (n+2)\alpha_0$$
$$+ (n-2) \cosh (n+4)\alpha_0\} - 2\{(\kappa+1)e^{-\alpha_0} + e^{-3\alpha_0}\}\delta_{n,1},$$
$$S_{C7} = (n+3)\{(n+3+\kappa) \cosh (n+2)\alpha_0 + (n+2) \cosh (n+4)\alpha_0\},$$
$$\bar{S}_{A1} = (n-2)(n-3) \cosh (n-2)\alpha_0,$$
$$\bar{S}_{A2} = n\{(n+1) \cosh (n-2)\alpha_0 + (n-1) \cosh (n+2)\alpha_0\} - 2 \cosh \alpha_0 \delta_{n,1},$$
$$\bar{S}_{A3} = (n+2)(n+3) \cosh (n+2)\alpha_0,$$
$$\bar{S}_{B1} = (n-3)\{(n-2) \cosh (n-4)\alpha_0 + (n-3-\bar{\kappa}) \cosh (n-2)\alpha_0\},$$
$$\bar{S}_{B2} = (n-1)\{(n+2) \cosh (n-4)\alpha_0 - 2(\bar{\kappa}-2) \cosh (n-2)\alpha_0$$
$$+ 4 \cosh n\alpha_0 + (n-1-\bar{\kappa}) \cosh (n+2)\alpha_0\} - (\bar{\kappa}+1)\delta_{n,2},$$
$$\bar{S}_{B3} = (n+1)\{(n+1+\bar{\kappa}) \cosh (n-2)\alpha_0 - 4 \cosh n\alpha_0 - 2(\bar{\kappa}-2) \cosh (n+2)\alpha_0$$
$$+ (n-2) \cosh (n+4)\alpha_0\} - 2\{(\bar{\kappa}+2) \cosh \alpha_0 + \cosh 3\alpha_0\}\delta_{n,1},$$
$$\bar{S}_{B4} = (n+3)\{(n+3+\bar{\kappa}) \cosh (n+2)\alpha_0 + (n+2) \cosh n\alpha_0\},$$

$$T_{A1} = (n-2)(n-3)\,\mathrm{e}^{-(n-2)\alpha_0},$$

$$T_{A2} = n\{(n+1)\,\mathrm{e}^{-(n-2)\alpha_0} + (n-1)\,\mathrm{e}^{-(n+2)\alpha_0}\} + 2\,\mathrm{e}^{\alpha_0}\delta_{n,1},$$

$$T_{A3} = (n+2)(n+3)\,\mathrm{e}^{-(n+2)\alpha_0},$$

$$T_{B1} = (n-3)\{(n-4)\,\mathrm{e}^{-(n-4)\alpha_0} + (n-3-\kappa)\,\mathrm{e}^{-(n-2)\alpha_0}\},$$

$$T_{B2} = (n-1)\{n\,\bar{\mathrm{e}}^{-(n-4)\alpha_0} - 2(\kappa-2)\,\mathrm{e}^{-(n-2)\alpha_0} + (n-1-\kappa)\,\mathrm{e}^{-(n+2)\alpha_0}\} + \{(\kappa+1) + 2\mathrm{e}^{-2\alpha_0}\}\delta_{n,2},$$

$$T_{B3} = (n+1)\{(n+1+\kappa)\,\mathrm{e}^{-(n-2)\alpha_0} + 2(\kappa-2)\,\mathrm{e}^{-(n+2)\alpha_0} + n\,\mathrm{e}^{-(n+4)\alpha_0}\} + 2\{(\kappa+2) + 3\,\mathrm{e}^{-3\alpha_0}\}\delta_{n,1},$$

$$T_{B4} = (n+3)\{(n+3+\kappa)\,\mathrm{e}^{-(n+2)\alpha_0} + (n+4)\mathrm{e}^{-(n+4)\alpha_0}\},$$

$$T_{C1} = (n-2)(n-3)\sinh(n-2)\alpha_0,$$

$$T_{C2} = n\{(n+1)\sinh(n-2)\alpha_0 + (n-1)\sinh(n+2)\alpha_0\},$$

$$T_{C3} = (n+2)(n+3)\sinh(n+2)\alpha_0,$$

$$T_{C4} = (n-3)\{(n-4)\sinh(n-4)\alpha_0 + (n-3-\kappa)\sinh(n-2)\alpha_0\},$$

$$T_{C5} = (n-1)\{n\sinh(n-4)\alpha_0 - 2(\kappa-2)\sinh(n-2)\alpha_0 + (n-1-\kappa)\sinh(n+2)\alpha_0\} + \{(\kappa+1) + 2\mathrm{e}^{-2\alpha_0}\}\delta_{n,2},$$

$$T_{C6} = (n+1)\{(n+1+\kappa)\sinh(n-2)\alpha_0 + 2(\kappa-2)\sinh(n+2)\alpha_0 + n\sinh(n+4)\alpha_0\} + 2\{(\kappa+2) + 3\,\mathrm{e}^{-3\alpha_0}\}\delta_{n,1},$$

$$T_{C7} = (n+3)\{(n+3+\kappa)\sinh(n+2)\alpha_0 + (n+4)\sinh(n+4)\alpha_0\},$$

$$\bar{T}_{A1} = (n-2)(n-3)\sinh(n-2)\alpha_0,$$

$$\bar{T}_{A2} = n\{(n+1)\sinh(n-2)\alpha_0 + (n-1)\sinh(n+2)\alpha_0\} + 2\sinh\alpha_0\delta_{n,1},$$

$$\bar{T}_{A3} = (n+2)(n+3)\sinh(n+2)\alpha_0,$$

$$\bar{T}_{B1} = (n-3)\{(n-4)\sinh(n-4)\alpha_0 + (n-3-\bar{\kappa})\sinh(n-2)\alpha_0\},$$

$$\bar{T}_{B2} = (n-1)\{n\sinh(n-4)\alpha_0 - 2(\bar{\kappa}-2)\sinh(n-2)\alpha_0 + (n-1-\bar{\kappa})\sinh(n+2)\alpha_0\} + 2\sinh 2\alpha_0\delta_{n,2}$$

$$\bar{T}_{B3} = (n+1)\{(n+1+\bar{\kappa})\sinh(n-2)\alpha_0 + 2(\bar{\kappa}-2)\sinh(n+2)\alpha_0 + n\sinh(n+4)\alpha_0\} + 2\{(\bar{\kappa}+2)\sinh\alpha_0 + 3\sinh 3\alpha_0\}\delta_{n,1},$$

$$\bar{T}_{B4} = (n+3)\{(n+3+\bar{\kappa})\sinh(n+2)\alpha_0 + (n+4)\sinh(n+4)\alpha_0\}.$$